Stochastic characterization of harmonic sections and a Liouville theorem

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Abstract

Let $P(M,G)$ be a principal fiber bundle and $E(M,N,G,P)$ be an associate fiber bundle. Our interested is to study harmonic sections of the projection π_E of E into M. Our first purpose is to give a stochastic characterization of harmonic section from M into E and a geometric characterization of harmonic sections with respect to its equivariant lift. The second purpose is to show a version of Liouville theorem for harmonic sections and to prove that section M into E is a harmonic section if and only if it is parallel.

Key words: harmonic sections; Liouville theorem; stochastic analisys on manifolds

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1 Introduction

Let $\pi_E : (E,k) \to (M,g)$ be a Riemannian submmersion and σ be a section of π_E, that is, $\pi_E \circ M = Id_M$. We know that $TE = VE \oplus HE$ such that $VE = \ker(\pi_{E*})$ and HE is the horizontal bundle ortogonal to VE. C. Wood has studied the harmonic sections in many context, see [16], [17], [18], [19], [20]. To recall, a harmonic sections is a minimal section for the vertical energy functional

$$E(\sigma) = \frac{1}{2}\int_M \|\mathbf{v}\sigma_*\|^2 vol(g),$$

where $\mathbf{v}\sigma_*$ is the vertical component of σ_*. Furthermore, in [16], Wood showed that σ is a minimizer of the vertical energy functional if

$$\tau_\sigma^v = \mathrm{tr}\nabla^{\mathrm{v}}\mathbf{v}\sigma_* = 0,$$

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where ∇^v is the vertical part of Levi-Civita connection on E, since π_E has totally geodesics fibers. Wood called σ a harmonic section if $\tau_\sigma^v = 0$.

In this work, we drop the Riemannian submersion condition of π_E and we mantain the fact that $TE = VE \oplus HE$ and that M is a Riemmanian Manifold. Let ∇^E be a symmetric connection on E, where E is not necessarily a Riemannian manifold. About these conditions we can define harmonic sections in the same way that Wood, only observing that ∇^v is vertical connection induced by ∇^E. There is no compatibility between ∇^E and Levi-Civita connection on M. The only assumption about ∇^E is that π_E has totally geodesic fibres.

Furthermore, we restrict the context of our study. Let $P(M, G)$ be a Riemannian G-principal fiber bundle over a Riemannian manifold M such that the projection π of P into M is Riemmanian submmersion. Suppose that P has a connection form ω. Let $E(M, N, G, P)$ be an associated fiber bundle of P with fiber N. It is well know that ω yields horizontal spaces on E. Our goal is to study the harmonic sections of projection π_E.

Let $F : P \to N$ be a differential map. We call F a horizontally harmonic map if $\tau_F \circ (H \otimes H) = 0$, where H is the horizontal lift from M into P associated to ω.

Let σ be a section of π_E. It is well know that there exists a unique equivariant lift $F_\sigma : P \to N$ associated to σ. Our first purpose is to give an stochastic characterization for the harmonic section σ and the horizontally harmonic map F_σ. From these stochastic characterizations we show that a section σ of π_E is harmonic section if and only if F_σ is a horizontally harmonic map. This result is an extension of Theorem 1 in [16]. As an example of application of stochastic characterization of harmonic sections, we will characterize the harmonic sections on tangent bundle with complete and horizontal lifts.

For our second purpose we consider $P(M, G)$ endowed with the Kaluza-Klein metric, M and G with the Brownian coupling property and N with the non-confluence property. About these conditions we show a version of Liouville Theorem and a version of result due to T. Ishiara in [7] to harmonic sections. As applications of our Liouville Theorem we can show the following. If we suppose that M is complete Riemmanian manifold with nonnegative Ricci curvature and its tangent bundle TM is endowed with the Sasaky metric, then the harmonic sections σ of π_{TM} are the 0-section. In the same way we can construct an ambient for Hopf fibrations, with Riemannian structure, such that harmonic sections are the 0-section.

2 Preliminaries

In this work we use freely the concepts and notations of P. Protter [13], E. Hsu [6], P. Meyer [11], M. Emery [3] and [5], W. Kendall [10] and S. Kobayashi and N. Nomizu [8]. We refer the reader to [1] for a complete survey about the objects of this section.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space which satisfies the usual hypothesis (see for example [3]). Our basic assumptions is that every stochastic process are continuos.

Definition 2.1 *Let M be a differential manifold. Let X be a process stochastic with valued in M. We call X a semimartingale if, for all f smooth on M, $f(X)$ is a real semimartingale.*

Let M be a differential manifold endowed whit symmetric connection ∇^M. Let X be a semimartingale in M and θ be a 1-form on M defined along X. Let $(x_1, \ldots, x_n)$ be a local coordinate system on M. We define the Itô integral of θ along X, locally, by

$$\int_0^t \theta d^{\nabla^M} X_s = \int_0^t \theta_i(X_s) dX_s^i + \frac{1}{2}\int_0^t \Gamma_{jk}^i(X_s)\theta_i(X_s) d[X^j, X^k]_s,$$

where $\theta = \theta_i dx^i$ with θ_i smooth functions and Γ_{jk}^i are the Cristoffel symbol of connection ∇^M. Let $b \in T^{(2,0)}M$ defined along X. We define the quadratic integral on M along X, locally, by

$$\int_0^t b\ (dX, dX)_s = \int_0^t b_{ij}(X_s) d[X^i, X^j]_s,$$

where $b = b_{ij} dx^i \otimes dx^j$ with b_{ij} smooth functions. Furthermore, we denote the Stratonovich integral by $\int_0^t \theta \delta X_s$.

Let M and N be differential manifolds endowed with symmetric connections ∇^M and ∇^N, respectively. Let $F : M \to N$ be a differential map and θ be a section of TN^*. We have the following geometric Itô formula:

$$\int_0^t \theta\ d^{\nabla^N} F(X_s) = \int_0^t F^*\theta\ d^{\nabla^M} X_s + \frac{1}{2}\int_0^t \beta_F^*\theta\ (dX, dX)_s, \tag{1}$$

where β_F is the second fundamental form of F (see [1] or [15] for the definition of β_F). It is well known that F is affine map if $\beta_F \equiv 0$.

Definition 2.2 *Let M be a differential manifold endowed with symmetric connection ∇^M. A semimartingale X with values in M is called a ∇^M-martingale if $\int_0^t \theta\ d^M X_s$ is a real local martingale for all $\theta \in \Gamma(TM^*)$.*

Definition 2.3 *Let M be a Riemannian manifold equipped with metric g. Let B be a semimartingale with values in M, we say that B is a g-Brownian motion in M if B is a ∇^g-martingale, where ∇^g is the Levi-Civita connection of g, and for any section b of $T^{(2,0)}M$ we have that*

$$\int_0^t b(dB, dB)_s = \int_0^t \mathrm{tr}\ \mathrm{b_{B_s}ds}. \tag{2}$$

From (1) and (2) we deduce the useful formula:

$$\int_0^t \theta d^{\nabla^N} F(B_s) = \int_0^t F^* \theta d^{\nabla^g} B_s + \frac{1}{2} \int_0^t \tau_F^* \theta_{B_s} ds, \tag{3}$$

where τ_F is the tension field of F.

From formula (2) and Doob-Meyer decomposition it follows that F is an harmonic map if and only if it sends g-Brownian motions to ∇^N-martingales.

Definition 2.4 *Let M be a differential manifold endowed with symmetric connection ∇^M. M has the non-confluence of martingales property if for every filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, M-valued martingales X and Y defined over Ω and every finite stopping time T such that*

$$X_T = Y_T \quad a.s. \text{ we have } X = Y \quad \text{over} \quad [0, T].$$

Example 2.1 *Let $M = V$ be a n-dimensional vector space with flat connection ∇^n. Let X and Y be V-valued martingales. Suppose that there are a stopping time τ with respect to $(\mathcal{F}_t)_{t \geq 0}$, $K > 0$ such that $\tau \leq K < \infty$ and $X_\tau = Y_\tau$. Then straightforward calculus shows that $X_t = Y_t$ for $t \in [0, \tau]$.*

Definition 2.5 *A Riemmanian manifold M has the Brownian coupling property if for all $x_0, y_0 \in M$ we can construct a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t; t \geq 0)$ and two Brownian motions X and Y, not necessarily independents, but both adapted to filtration such that*

$$X_0 = x_0, Y_0 = y_0$$

and

$$\mathbb{P}(X_t = Y_t \text{ for some } t \geq 0) = 1.$$

The stopping time $T(X, Y) = \inf\{t > 0; X_t = Y_t\}$ is called coupling time.

Example 2.2 *Let M be a complete Riemannian manifold. In [9], W. Kendall has showed that if M is compact or M has nonnegative Ricci curvature then M has the Brownian coupling property.*

Let M be a Riemmanian manifold with metric g. Consider X and Y two g-Brownian motion in M which satisfies the Brownian coupling property and $X_0 = x, Y_0 = y$, where $x, y \in M$. Denote by $T(X,Y)$ their coupling time. The process $\bar{Y}$ is defined by

$$\bar{Y}_t = \begin{cases} Y_t & , \quad t \leq T(X,Y) \\ X_t & , \quad t \geq T(X,Y). \end{cases} \tag{4}$$

It is imediatelly that $\bar{Y}_0 = y_0$.

Proposition 2.1 *Let M be a Riemannian manifold with metric g. Suppose that M has the Brownian coupling property. Let X, Y be two g-Brownian motions in M which satisfies the Brownian coupling property. Then the process $\bar{Y}$ is a g-Brownian motion in M.*

Proof: It is a straightforward proof from definition of Brownian motion. $\square$

3 Harmonic sections

Let $P(M,G)$ be a principal fiber bundle over M and $E(M,N,G,P)$ be an associate fiber bundle to $P(M,G)$. We denote the canonical projection from $P \times N$ into E by μ, namely, $\mu(p,\xi) = p \cdot \xi$. For each $p \in P$, we have the map $\mu_p : N \to E$ defined by $\mu_p(\xi) = \mu(p,\xi)$. Let $\sigma : E \to M$ be a section of projection π_E, that is, $\pi_E \circ \sigma = Id_M$. There exists a unique equivariant lift $F_\sigma : P \to N$ associated to σ which is defined by

$$F_\sigma(p) = \mu_p^{-1} \circ \sigma \circ \pi(p). \tag{5}$$

The equivariance property of F_σ is given by

$$F_\sigma(p \cdot g) = g^{-1} \cdot F_\sigma(p), \quad g \in G.$$

Let us endow P and M with Riemmanian metrics k and g, respectively, such that $\pi : (P,k) \to (M,g)$ is a Riemmanian submmersion. Let ω be a connection form on P. We observe that the connection form ω yields a horizontal structure on E, that is, for each $b \in E$, $T_bE = V_bE \oplus H_bE$, where $V_bE := \mathrm{Ker}(\pi_{Eb*})$ and H_bE is the horizontal subspace done by ω on E (see for example [8], pp.87). We denote by $\mathbf{v} : TE \to VE$ and $\mathbf{h} : TE \to HE$ the vertical and horizontal projection, respectively.

Let ∇^M denote the Levi-Civita connection on M and ∇^E be a symmetric connection on E. We are interested in connections ∇^E such that the projection π_E from E into M has totally geodesic fibres.

We denote by ∇^v the vertical component of connection ∇^E on TE, that is, for X a vector field and V a vertical vector field on E we have

$$\nabla^v_X V = \mathbf{v}\nabla^E_X V.$$

Let us denote ∇^x the induced connection of ∇^v over fiber $\pi_E^{-1}(x)$ for all $x \in M$. We endow N with a connection ∇^N such that, for each $p \in P$, μ_p is an affine map over its image, the fiber $\pi_E^{-1}(x)$ with $\pi(p) = x$.

As π_E has totally geodesics fibres we have, for each $p \in P$, $\beta^v_{\mu_p} = \beta^x_{\mu_p}$, where

$$\beta^v_{\mu_p} = \nabla^v \circ \mu_{p*} - \mu_{p*}\nabla^N, \quad \beta^x_{\mu_p} = \nabla^x \circ \mu_{p*} - \mu_{p*}\nabla^N$$

and $\pi(p) = x$. Since μ_p is affine map, for each $p \in P$, we conclude that $\beta^v_{\mu_p} \equiv 0$. In summary we have the following

Lemma 3.1 *Let π_E be the projection from E into M. Let ∇^E be a symmetric connection on E such that π_E has totally geodesic fibres. Suppose that N is endowed with a connection such that, for each $p \in P$, μ_p is an affine map. Then $\beta^v_{\mu_p} \equiv 0$ for all $p \in P$.*

Let σ be a section of π_E. Write $\sigma_* = \mathbf{v}\sigma_* + \mathbf{h}\sigma_*$, where $\mathbf{v}\sigma_*$ and $\mathbf{h}\sigma_*$ are the vertical and the horizontal component of σ_*, respectively. The second fundamental form for $\mathbf{v}\sigma_*$ is defined by

$$\beta^v_\sigma = \bar{\nabla}^v \circ \mathbf{v}\sigma_* - \mathbf{v}\sigma_* \circ \nabla^M,$$

where $\bar{\nabla}^v$ is the induced connection on $\sigma^{-1}E$. The vertical tension field is given by

$$\tau^v_\sigma = \mathrm{tr}\beta^{\mathrm{v}}_\sigma.$$

In the following we extend the definition given by C. M. Wood [17] of harmonic section.

Definition 3.1 *1. A section σ of π_E is called harmonic section if $\tau^v_\sigma = 0$;*
2. A differential map $F : P \to N$ is called horizontally harmonic if $\tau_F \circ (H \otimes H) = 0$, where H is horizontal lift from M into P.

Definition 3.2 *1. Let θ be a 1-form on E. We call θ a vertical form if $\theta \in VE^*$, the adjoint of vertical bundle VE.*
2. A E-valued semimartingale X is called a vertical martingale if, for every vertical form θ on E, $\int_0^t \theta d^{\nabla^v} X_s$ is a real local martingale.

Now, we give a characterization of harmonic section in the context that we are working.

Theorem 3.1 *Let $E(M, N, G, P)$ be an associated fiber bundle to principal fiber bundle $P(M, G)$, where (M, g) is a Riemannian manifold. Let us endow E with a symmetric connection on ∇^E. Then a section σ of π_E is harmonic section if and only if, for every g-Brownian motion B in M, $\sigma(B)$ is a vertical martingale.*

Proof: Let B be a g-Brownian motion in M and θ be a vertical form on E. By formula (3),

$$\int_0^t \theta \; d^{\nabla^v}\sigma(B_s) = \int_0^t \sigma^*\theta \; d^{\nabla^M} B_s + \frac{1}{2}\int_0^t \tau_\sigma^{v*}\theta(B_s)\; ds.$$

We observe that $\int \sigma^*\theta \; d^{\nabla^M} B_s$ is a real local martingale. Since B and θ are arbitraries, Doob-Meyer decomposition assure that $\int_0^t \theta \; d^{\nabla^v}\sigma(B_s)$ is real local martingale if and only if τ_σ^v vanishes. From definitions of vertical martingale and harmonic section we conclude the proof. □

Remark 1 In this Theorem we observe that we do not need the hypotesis of principal fiber bundle be a Riemannian manifold and $\pi : P \to M$ be a Riemannian submmersion. It is possible to give this characterization in general way for a submersion.

Before we follow we need to give the definition of horizontal semimartingale in the fiber bundle P. Furthermore, to prove our next result it is necessary to writer a horizontal semimartingale in P as coordinates of a local coordinate system of P .

Definition 3.3 *Let Y be a semimartingale in P. We say that Y is a horizontal martingale if $\int_0^t \omega\delta Y_s = 0$, where ω is the connection 1-form on P.*

It is clear that if Y is a horizontal semimartingale in P then δY_t in HP, the horizontal bundle given by the connection 1-form ω. Let $(U \times V, x^i, v^l)$ be a local coordinate system in P. We wish to describe Y in this coordinates. For this end, firstly, we need describe the generators for H_pP, with $p \in U \times V$. In fact, we observe that (U, x^i) is a local coordinate system in M, so we denote the coordinate vector fields $\partial/\partial x^i$ by D_i, $i = 1, \ldots, n$. Let $H_p : T_{\pi(p)}M \to T_pP$, $p \in P$, the family of linear isomorphism yielded by ω. We recall that this family is called horizontal lift from M to P associated to ω. Because D_i, $i = 1, \ldots n$, is a local frame on U, we have that HD_i is

a horizontal local frame on $U \times V$. Let us denote $\bar{D}_l$ the coordinate vector field in VP, the vertical fiber bundle. To obtain or goal is sufficient describe the vector HD_i in terms of D_i and $\bar{D}_l$. In fact, for $i = 1, \ldots, n$ a simple account shows that

$$HD_i = D_i - \omega_i^l \bar{D}_l, \quad i = 1, \ldots n \text{ and } \; l = 1, \ldots m$$

where ω_l^i are coordinates of $\omega(D_i)$ in the Lie algebra $\mathfrak{g}$ and m is the dimension of Lie group G that acts in P. It follows that

$$\delta Y_t = \delta Y_t^i HD_i = \delta Y_t^i D_i - \delta Y_t^i \omega_i^l \bar{D}_l.$$

From this we conclude that Y has the following coordinates in $U \times V$

$$Y_t = (Y_t^1, \ldots, Y_t^n, -\int_0^t \delta Y_s^i \omega_i^1, \ldots, -\int_0^t \delta Y_s^i \omega_i^m) \tag{6}$$

We use this fact in the proof of our next Lemma, which is a key in the demonstration of our after Proposition.

Lemma 3.2 *Let X_t be a E-valued semimartingale such that $X_t = \mu(Y_t, \xi_t)$, where Y_t is a horizontal P-semimartingale and ξ_t is a N-semimartingale. Then*

$$\int_0^t \theta d^{\nabla^v} X_s = \int_0^t \theta_\alpha d^{\nabla^v} \mu_{Y_s}(\xi_s)$$

for all vertical form θ on E, where $\mu_{Y_t}(\xi_t)$ is the vertical semimartingale in E associated to X.

Proof: Let $(U \times V \times W, x^i, v^l, \nu^\alpha)$ be a local coordinate system in $P \times N$. Thus (U, x^i), (V, v^l) and (W, ν^α) are local coordinate systems in M, G and N, respectively. Let X_t be a E-semimartingale as assumption and θ a vertical form. By definition of Itô integral,

$$\int_0^t \theta d^{\nabla^v} X_s = \int_0^t \theta_\alpha(X_s) dX_s^\alpha + \frac{1}{2}\int_0^t \Gamma_{\beta\gamma}^\alpha(X_s)\theta_\alpha(X_s) d[X^\beta, X^\gamma]_s, \tag{7}$$

where $X_t^\alpha = \nu^\alpha(X_t)$. It is sufficient to prove that $dX_t^\alpha = d(\mu_{Y_t}\xi_t)^\alpha$ for all $\alpha = 1, \ldots, r$. Here, we observe that ν^α, $\alpha = 1, \ldots, n$, are vertical local coordinates in E and it is not dependent of local coordinate system, because Itô integral do not depend. Let ω be a connection form on P and $(U \times V, x^i, v^l)$ be a local coordinate system in P. Since Y is a horizontal semimartingale in P, from (6) we see that

$$Y = (Y^1, \ldots, Y^n, -\int dY^i \omega_i^1, \ldots, -\int dY^i \omega_i^m).$$

Therefore, the process (Y_s, ξ_s) can be written, in coordinates, as

$$(Y^1, \dots, Y^n, -\int dY^i\omega_i^1, \dots, -\int dY^i\omega_i^n, \xi^1, \dots, \xi^r).$$

Let us denote $D_i = \partial/\partial x^i$, $\bar{D}_l = \partial/\partial v^l$ and $\tilde{D}_\beta = \partial/\partial \nu^\beta$, for $i = 1, \dots, n$, $l = 1, \dots, m$ and $\beta = 1, \dots, r$. In the sequence, we consider the functions applied at (Y_t, ξ_t). Applying the Itô formula in $v^\alpha\mu(Y_t, \xi_t)$ we obtain

$$\begin{aligned}
dv^\alpha\mu(Y_t, \xi_t) &= D_i(v^\alpha\mu)dY_t^i + \bar{D}_l(v^\alpha\mu)d(\int dY^i\omega_l^i)_t + \tilde{D}_\beta(v^\alpha\mu)d\xi_t^\beta \\
&+ \frac{1}{2}D_jD_i(v^\alpha\mu)d[Y^i, Y^j]_t + \frac{1}{2}\bar{D}_lD_i(v^\alpha\mu)d[-\int dY^j\omega_l^j, Y^j]_t \\
&+ \frac{1}{2}D_i\bar{D}_l(v^\alpha\mu)d[Y^j, -\int dY^j\omega_l^j]_t + \frac{1}{2}\bar{D}_k\bar{D}_l(v^\alpha\mu)d[-\int dY^j\omega_k^j, -\int dY^j\omega_l^j]_t \\
&+ \frac{1}{2}D_i\tilde{D}_\beta(v^\alpha\mu)d[Y^i, \xi^\beta]_t + \frac{1}{2}\tilde{D}_\beta D_i(v^\alpha\mu)d[\xi^\beta, Y^i]_t \\
&+ \frac{1}{2}\tilde{D}_\beta\bar{D}_l(v^\alpha\mu)d[\xi^\beta, -\int dY^j\omega_l^j]_t + \frac{1}{2}\bar{D}_l\tilde{D}_\beta(v^\alpha\mu)d[-\int dY^j\omega_l^j, \xi^\beta]_t \\
&+ \frac{1}{2}\tilde{D}_\beta\tilde{D}_\gamma(v^\alpha\mu)d[\xi^\beta, \xi^\gamma]_t. \qquad (8)
\end{aligned}$$

Now, we observe that $D_i - \omega_i^l\bar{D}_l$ is a horizontal vector field on P. Hence, for $\xi \in N$, $\mu_{\xi*}(D_i - \omega_i^l\bar{D}_l)$ is a horizontal vector field on E. However dv^α is a vertical form on E for all $\alpha = 1, \dots, r$. It follows that $dv^\alpha\mu_{\xi*}(D_i - \omega_i^l\bar{D}_l) \equiv 0$. Applying these observations in the equality above we conclude that

$$dv^\alpha\mu(Y_t, \xi_t) = \tilde{D}_\beta(v^\alpha\mu)d\xi_t^\beta + \frac{1}{2}\tilde{D}_\beta\tilde{D}_\gamma(v^\alpha\mu)d[\xi^\beta, \xi^\gamma]_t.$$

It follows that

$$dX_t^\alpha = dv^\alpha\mu(Y_t, \xi_t) = dv^\alpha\mu_{Y_t}\xi_t = d(\mu_{Y_t}\xi_t)^\alpha.$$

Applying this equality in (7) we conclude that

$$\int_0^t \theta d^{\nabla^v} X_s = \int_0^t \theta d^{\nabla^v}\mu_{Y_s}\xi_s.$$

$\square$

Remark 2 In the demonstration above one could think that the local coordinate system is relevant, but if we see (8) as the second vector field of $P \times N$-semimartingale (Y_t, ξ_t) applied to function ν^α then from Schwartz principle the second vector field (Y_t, ξ_t) do not depend of local coordinate system.

Remark 3 The Lemma above shows the strength of Itô integral and Itô formula. One can try to use the Stratonovich-Itô conversion formula (see for example [2]) to conclude the same Lemma above. For this end, one will need some restriction about connection ∇^E. In fact, we need that the vertical part of $\nabla^E_V X$ and $\nabla^E_X V$ vanishes, where V and X are vertical and horizontal vector fields, respectively, on E.

Now, we relate the geometric and stochastic concepts of harmonic section and horizontally harmonic map.

Proposition 3.2 *Let $P(M,G)$ be a Riemannian principal fiber bundle endowed with a connection form ω and M a Riemannian manifold such that the projection π of P into M is a Riemannian submmersion. Let $E(M,N,G,P)$ be an associated fiber to P endowed with a symmetric connection ∇^E such that the projection π_E has totally geodesic fibres. Moreover, suppose that N has a connection ∇^N such that μ_p is an affine map for each $p \in P$. Then*

(i) *a E-valued semimartingale X is vertical martingale if and only if $\mu_Y^{-1} \circ X$ is a ∇^N- martingale in N, where $Y = \pi_E(X)^h$ is the horizontal lift of $\pi_E(X)$ to P;*

(ii) *a equivariant lift F_σ associated to σ, σ a section of π_E, is horizontally harmonic map if and only if, for every horizontal Brownian motion B^h in P, $F_\sigma(B^h)$ is a ∇^N-martingale.*

Proof: **(i)** Let X be a semimartingale in E and θ be a vertical form on E. Let us denote $\xi = \mu_Y^{-1} \circ X$. As $X = \mu(Y,\xi)$ we have, by Lemma 3.2,

$$\int_0^t \theta d^{\nabla^v} X_s = \int_0^t \theta d^{\nabla^v} \mu_{Y_s} \xi_s.$$

By geometric Itô formula (1),

$$\int_0^t \theta d^{\nabla^v} X_s = \int_0^t \mu_{Y_s}^* \theta d^{\nabla^N} \xi_s + \frac{1}{2} \int \beta^{v*}_{\mu_{Y_s}} \theta(d\xi_s, d\xi_s).$$

From Lemma 3.1 we deduce that

$$\int_0^t \theta d^{\nabla^v} X_s = \int_0^t \mu_{Y_s}^* \theta d^{\nabla^N} \xi_s.$$

So we conclude that $\int_0^t \theta d^{\nabla^v} X_s$ is local martingale if and only if $\int_0^t \mu_{Y_s}^* \theta d^{\nabla N} \xi_s$ is too, and proof is complete.

(ii) Let B be a g-Brownian motion in M and B^h be a horizontal Brownian motion in P, that is,

$$dB^h = H_B dB, \tag{9}$$

where H is the horizontal lift of M to P. Set $\theta \in \Gamma(TN^*)$. By geometric Itô formula (1),

$$\int_0^t \theta \ d^{\nabla^N} F_\sigma(B_s^h) = \int_0^t F_\sigma^* \theta \ d^{\nabla^P} B_s^h + \int_0^t \beta_{F_\sigma}^* \theta (dB^h, dB^h)_s.$$

From (9) we see that

$$\int_0^t \theta \ d^{\nabla^N} F_\sigma(B_s^h) = \int_0^t H^* F_\sigma^* \theta \ d^{\nabla^M} B_s + \int_0^t \beta_{F_\sigma}^* \theta (H_B dB, H_B dB)_s.$$

As B is Brownian motion we have

$$\int_0^t \theta \ d^{\nabla^N} F_\sigma(B_s^h) = \int_0^t H^* F_\sigma^* \theta \ d^{\nabla^M} B_s + \int_0^t (\tau_{F_\sigma}^H)^* \theta(B_s) ds,$$

where $\tau_{F_\sigma}^H = \tau_{F_\sigma} \circ (H \otimes H)$. Since θ and B are arbitraries, Doob-Meyer decomposition shows that $\int_0^t \theta d^{\nabla^N} F_\sigma(B_s^h)$ is real local martingale if and only if $\tau_{F_\sigma}^H$ vanishes. From definitions of martingale and horizontally harmonic map we conclude the proof. $\square$

Remark 4 In the equation (9) we can see the necessary of hypotesys of Riemannian submmersion over $\pi : P \to M$. In fact, it is well-know that the horizontal Brownian motion is defined as Stratonovich stochastic equation, see for example [14]. However the Corollary 16 in [4] shows that Stratonovich and Itô differential equations are equivalent because the horizontal lift of geodesic in M is a geodesic in P, since π is a Riemannian submmersion.

Now we give an extension of the characterization of harmonic sections obtained by C.M. Wood, see Theorem 1 in [17].

Theorem 3.3 *Under the hypotheses of Proposition 3.2, a section σ of π_E is harmonic section if and only if F_σ is horizontally harmonic map.*

Proof: Let B be a arbitrary g-Brownian motion in M and B^h be a horinzontal lift of B in P, see equation (9).

Suppose that σ is a harmonic section. Theorem 3.1, shows that $\sigma(B)$ is a vertical martingale. But $\mu_{B^h}^{-1} \circ \sigma(B)$ is a ∇^N-martingale, which follows from Proposition 3.2, item (i). Since $F_\sigma(B^h) = \mu_{B^h}^{-1} \circ \sigma \circ \pi(B^h)$, it follows

that $F_\sigma(B^h)$ is a ∇^N-martingale. Finally, Proposition 3.2, item (ii), shows that F_σ is horizontally harmonic map.

Conversely, suppose that F_σ is a horizontally harmonic map. Proposition 3.2, item (ii), shows that $F_\sigma(B^h)$ is a ∇^N-martingale. Since $F_\sigma(B^h) = \mu_{B^h}^{-1} \circ \sigma \circ \pi(B^h)$, it follows that $\mu_{B^h}^{-1} \circ \sigma(B)$ is a ∇^N-martingale. From Proposition 3.2, item (i), we see that $\sigma(B)$ is a vertical martingale. We conclude from Theorem 3.1 that σ is a harmonic section. $\square$

4 A Liouville theorem for harmonic sections

We begin this section defining the Kaluza-Klein metric on $P(M,G)$. Let $P(M,G)$ be a principal fiber bundle endowed with a connection form ω, M be a Riemannian manifold with a metric g and h be a bi-invariant metric on G. The Kaluza-Klein metric is defined by

$$k = \pi^* g + \omega^* h. \tag{10}$$

From now on $P(M,G)$ is endowed with the Kaluza-Klein metric.

We will denote by d_P and d_G the Riemannian distance of P and G, respectively.

Lemma 4.1 *Let $P(M,G)$ be a principal fiber bundle whit a Kaluza-Klein metric k, where g is the Riemannian metric on M and h is the bi-invariant metric on G associated to k. The following assertions are holds:*

(i) *Let $\tau : [0,1] \to P$ be a differential curve such that $\tau(t) = u \cdot \mu(t)$ with $\tau(0) = u$ and $\mu(t) \in G$, then*

$$\int_0^1 k(\dot\tau(t), \dot\tau(t))^{\frac{1}{2}} dt = \int_0^1 h(\dot\mu(t), \dot\mu(t))^{\frac{1}{2}} dt.$$

(ii) *Let $\tau : [0,1] \to P$ be a differential curve. If γ is a curve in M and if μ is a curve in G such that $\tau = \gamma(t)^h \cdot \mu(t)$, then*

$$\int_0^1 k(\dot\tau(t), \dot\tau(t))^{\frac{1}{2}} dt \leq \int_0^1 g(\dot\gamma(t), \dot\gamma(t))^{\frac{1}{2}} dt + \int_0^1 h(\dot\mu(t), \dot\mu(t))^{\frac{1}{2}} dt$$

(iii) *Let $x \in M$ and $u, v, w \in \pi^{-1}(x)$. If a, b are points in G such that $v = u \cdot a$ and $w = u \cdot b$, then*

$$d_P(v, w) = d_G(a, b).$$

Proof: **(i)** and **(ii)** The proofs are straightforward.
(iii) Let $\tau : [0,1] \to P$ be a differential curve such that $\tau(0) = v$ and $\tau(1) = w$. Consider a curve γ in M such that $\pi(\tau) = \gamma$. There exists a differential curve μ in G such that $\mu(0) = a$, $\mu(1) = b$ and $\tau = \gamma^h \cdot \mu$. We observe that $\gamma(0) = x$ and $\gamma(1) = x$. This gives $\int_0^1 g(\dot\gamma(t), \dot\gamma(t))^{\frac{1}{2}} dt = 0$. Thus from item (i) and item (ii) we conclude that

$$\int_0^1 k(\dot\tau(t), \dot\tau(t))^{\frac{1}{2}} dt = \int_0^1 h(\dot\mu(t), \dot\mu(t))^{\frac{1}{2}} dt.$$

Therefore it is only necessary to consider vertical curves. It follows that $d_P(v,w) = d_P(u \cdot a, u \cdot b) = d_G(a,b)$, by definition of Riemmanian distance. $\square$

Theorem 4.1 *Let $P(M,G)$ be a principal fiber bundle equipped with Kaluza-Klein metric and $E(M,N,G,P)$ be an associated fiber to P. Let ∇^E and ∇^N be connetions on E and N, respectively, such that the projection π_E has totally geodesic fibres and μ_p is an affine map for each $p \in P$. Moreover, if N has the non-confluence martingales property and if M and G have the Brownian coupling property, then*

(i) *a section σ of π_E is harmonic section if and only if F_σ is constante map;*

(ii) *the left action of G into N has a fix point if there exists a harmonic section σ of π_E;*

(iii) *a section σ of π_E is harmonic section if and only if σ is parallel.*

Proof: **(i)** We first suppose that F_σ is a constante map. Then it is immediately that $\tau^v_\sigma = 0$, so σ is harmonic section.

Conversely, the proof will be divided into two parts. Firstly, we found a suitable stopping time τ. After, we use τ to prove that F_σ is constant over P.

Choose $x, y \in M$ arbitraries. By assumption about M, there exists two g-Brownian motion X and Y in M such that $X_0 = x$ and $Y_0 = y$, which satisfy the Brownian coupling property. Consequently, the coupling time $T(X,Y)$ is finite. Proposition 2.1 now assures that the process

$$\bar{Y}_t = \begin{cases} Y_t & , \quad t \leq T(X,Y) \\ X_t & , \quad t \geq T(X,Y) \end{cases} \tag{11}$$

is a g-Brownian motion in M.

Let $a, b \in G$ be arbitraries points. Since G has the Brownian coupling property, we have two h-Brownian motion μ and ν in G such that $\mu_0 = a$, $\nu_0 = b$. Moreover, there is a finite coupling time $T(\mu, \nu)$. But the process

$$\bar{\nu_t} = \begin{cases} \nu_t & , \quad t \leq T(\mu,\nu) \\ \mu_t & , \quad t \geq T(\mu,\nu) \end{cases} \tag{12}$$

is a h-Brownian motion in G, which follows from Proposition 2.1.

Set $u, v \in P$ such that $\pi(u) = x$ and $\pi(v) = y$. Consider two horizontal Brownian motion X^h and $\bar{Y}^h$ in P such that $X_0^h = u$ and $\bar{Y}_0^h = v$. Define $\tau = T(X,Y) \vee T(\mu,\nu)$. We claim that

$$X_t^h \cdot \mu_t = \bar{Y}_t^h \cdot \bar{\nu}_t, \text{ a.s. } \forall \;\; t \geq \tau. \tag{13}$$

In fact, we need consider two cases. First, suppose that $T(X,Y) \leq T(\mu,\nu)$. For all $t \geq T(\mu,\nu)$ we have

$$d_P(X_t^h \cdot \mu_t, \bar{Y}_t^h \cdot \bar{\nu}_t) = d_P(X_t^h \cdot \mu_t, \bar{Y}_t^h \cdot \mu_t) = d_P(R_{\mu_t} X_t^h, R_{\mu_t} \bar{Y}_t^h).$$

Since k is the Kaluza-Klein metric, it follows that

$$d_P(X_t^h \cdot \mu_t, \bar{Y}_t^h \cdot \bar{\nu}_t) = d_M(X_t, \bar{Y}_t).$$

From (11) we conclude that (13) is satisfied for all $t \geq T(\mu,\nu)$.

In the other side, suppose that $T(X,Y) \geq T(\mu,\nu)$. For all $t \geq T(X,Y)$, Lemma 4.1, item (iii), assures that

$$d_P(X_t^h \cdot \mu_t, \bar{Y}_t^h \cdot \bar{\nu}_t) = d_P(X_t^h \cdot \mu_t, X_t^h \cdot \bar{\nu}_t) = d_G(\mu_t, \bar{\nu}_t).$$

From (12) we conclude that (13) is satisfied for all $t \geq T(X,Y)$.

Setting $t \geq \tau$ we obtain $F_\sigma(X_t^h \cdot \mu_t) = F_\sigma(\bar{Y}_t^h \cdot \bar{\nu}_t)$. Since F_σ is equivariant by right action, $\mu_t^{-1} \cdot F_\sigma(X_t^h) = \bar{\nu}_t^{-1} \cdot F_\sigma(\bar{Y}_t^h)$. Because $\mu_t = \bar{\nu}_t$ for $t \geq \tau$, we conclude that $F_\sigma(X_t^h) = F_\sigma(\bar{Y}_t^h)$.

Since σ is a harmonic section, from Theorem 3.3 we see that F_σ is a horizontally harmonic map. Proposition 3.2 now shows that $F_\sigma(X_t^h)$ and $F_\sigma(\bar{Y}_t^h)$ are ∇^N-martingales in N. Since N has non-confluence martingales property,

$$F_\sigma(X_0^h) = F_\sigma(\bar{Y}_0^h).$$

It follows immediately that $F_\sigma(u) = F_\sigma(v)$. Consequently, F_σ is a constant map.

(ii) Let σ be a harmonic section of π_E. From item (i) there exists $\xi \in N$ such that $F_\sigma(p) = \xi$ for all $p \in P$. We claim that ξ is a fix point. In fact, set $a \in G$. From equivariant property of F_σ we deduce that

$$a \cdot \xi = a \cdot F_\sigma(p) = F_\sigma(p \cdot a^{-1}) = \xi.$$

(iii) Let σ be a section of π_E. Suppose that σ is parallel. Then $\sigma_*(X)$ is horizontal for all $X \in TM$ (see for example [8], pp.114). This gives $\mathbf{v}\sigma_*(X) = 0$. Then it is clear, by definition, that σ is harmonic section.

Suppose that σ is a harmonic section. From item (i) it follows that there exists $\xi \in N$ such that $F_\sigma(p) = \xi$ for all $p \in P$. By definition of equivariant lift,

$$\sigma(x) = \sigma \circ \pi(p) = \mu(p, \xi) = \mu_\xi(p), \quad \pi(p) = x,$$

where μ_ξ is an application from P into E. Let $v \in T_xM$ and let $\gamma(t)$ be a curve in M such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Then

$$\sigma_*(v) = \left.\frac{d}{dt}\right|_0 \sigma \circ \gamma(t) = \left.\frac{d}{dt}\right|_0 \mu_\xi \circ \gamma^h(t) = \mu_{\xi *}(\dot{\gamma}^h(0)),$$

where γ^h is the horizontal lift of γ into P. Since $\dot{\gamma}^h(0)$ is horizontal vector in P, so is $\mu_{\xi *}(\dot{\gamma}^h(0))$ in E (see for example [8], pp.87). Therefore $\sigma_*(v)$ is horizontal vector. So we conclude that σ is parallel. $\square$

5 Examples

In this section, we will give two applications. First, we will use the stochastic characterization of harmonic section, see Theorem 3.1, to give the harmonic sections on tangent bundle with complete and horizontal lifts. After, from Theorem 4.1 we will show that about geometric conditions the unique harmonic section on Tangent bundle with Sasaki metric is null. Finally, we will work with Hopf fibrations and harmonic sections.

Tangent Bundle with Complete and Horizontal lifts

Let M be a Riemmanian manifold and TM its tangent bundle. Let ∇^M be a symmetric connection on M. It is clear that TM is an associated fiber bundle to orthonormal frame bundles OM, which has, naturally, a Kaluza-Klein metric. Also, it is possible to prolong ∇^M to a connection on TM. A two well known ways are the complete lift ∇^c and horizontal lift ∇^h (see

[21] for the definitions of ∇^c and ∇^h). Let X, Y be vector fields on M, so ∇^h satisfies the following equations:

$$\begin{array}{rclrcl}
\nabla^c_{X^V} Y^V & = & 0 & \nabla^h_{X^V} Y^V & = & 0 \\
\nabla^c_{X^V} Y^H & = & 0 & \nabla^h_{X^V} Y^H & = & 0 \\
\nabla^c_{X^H} Y^V & = & (\nabla_X Y)^V & \nabla^h_{X^H} Y^V & = & (\nabla_X Y)^V \\
\nabla^c_{X^H} Y^H & = & (\nabla_X Y)^H + \gamma(R(-,X)Y, & \nabla^h_{X^H} Y^H & = & (\nabla_X Y)^H,
\end{array} \tag{14}$$

where $R(-,X)Y$ denotes a tensor field W of type (1,1) in M such that $W(Z) = R(Z,X)Y$ for any $Z \in T^{(0,1)}(M)$, and γ is a lift of tensor, which is defined at page 12 in [21].

First, we observe that the vertical connection ∇^v is the same one for complete and horizontal lifts. In particular, for U, V vertical vector fields on TM we see that $\nabla^v_U V \equiv 0$.

We wish to prove the following characterization of harmonic sections for complete and horizontal lifts.

Proposition 5.1 *Let M be a Riemannian manifold, ∇^M the Levi-Civita connection and TM its tangent bundle. Let σ be a section of π_{TM}. Then*

1. *If TM is endowed with ∇^c, then σ is a harmonic section with respect to ∇^c if and only if $\mathbf{v}\sigma_* \equiv 0$.*

2. *If TM is endowed with ∇^h, then σ is a harmonic section with respect to ∇^h if and only if $\mathbf{v}\sigma_* \equiv 0$.*

Proof: Let σ be a section of π_{TM} such that σ is a harmonic section with respect to ∇^c or ∇^h. Let $(U \times V, x^i, v^\alpha)$ be a local coordinate system on TM. We claim that $\sigma^\alpha = v^\alpha \circ \sigma$ is a constant function. In fact, for every Brownian motion B_t in M, Theorem 3.1 assures that $\sigma(B_t)$ is a vertical martingale in TM. In particular, suppose that $B_0 = x \in U$ and denote $\tau = inf\{t : B_t(\omega) \notin U, \forall\, \omega \in \Omega\}$, that is, the first exit time from U. Since the vertical connection ∇^v on TM is the same to ∇^c and ∇^h, from definition of Itô stochastic we have

$$\int_0^t dv^\alpha d^{\nabla^v} \sigma(B_s) = \int_0^t d\sigma^\alpha(B_s) + \frac{1}{2}\int_0^t \Gamma^\alpha_{\beta\gamma}(\sigma_s) d[\sigma^\beta(B), \sigma^\gamma(B)]_s.$$

where $0 \leq t \leq \tau$. By definition, $\nabla^v_U V \equiv 0$ for U, V vertical vector fields on TM. Thus $\Gamma^\alpha_{\beta\gamma} = 0$ for $\beta, \gamma = 1 \ldots n$, where n is the dimension of M. Therefore

$$\int_0^t dv^\alpha d^{\nabla^v} \sigma(B_s) = \int_0^t d\sigma^\alpha(B_s) = \sigma^\alpha(B_t)$$

Being $\sigma(B_t)$ a vertical martingale, it follows that $\sigma^\alpha(B_t)$ is a real local martingale. Applying the expectation at $\sigma^\alpha(B_t)$ we obtain

$$\mathbb{E}(\sigma^\alpha(B_t)) = \mathbb{E}(\sigma^\alpha(B_0)) = \sigma^\alpha(B_0).$$

From this we conclude, almost sure, that $\sigma^\alpha(B_t)$ is constant. Since B_t is an arbitrary Brownian motion, it follows that σ^α is a constant function.

To prove 1. and 2., it is only necessary to observe that $\mathbf{v}\sigma_*$, in any local coordinate system $(U \times V, x^i, v^\alpha)$, is written in function of σ^α and to use the conclusion above. □

Tangent bundle with Sasaki metric

Let M be a complete Riemannian manifold which is compact or has non-negative Ricci curvature. Let OM be the ortonormal frame bundle endowed whit the Kaluza-Klein metric. Let TM be the tangent bundle equipped with the Sasaki metric g_s. Thus π_E is a Riemannian submersion with totally geodesic fibers and, for each $p \in P$, μ_p is a isometric map (see for example [12]). From these assumptions and Examples 2.1 and 2.2 it follows that the hypotheses of Theorem 4.1 are satisfied.

Proposition 5.2 *Under conditions stated above, if σ is a harmonic section of π_{TM}, then σ is the 0-section.*

Proof: Let σ be a harmonic section of π_{TM}. By Theorem 4.1, item (i), there exists $\xi \in N$ such that $F_\sigma(u) = \xi$ for all $u \in P$. Moreover, by item (ii) ξ is a fix point of left action of $O(n, \mathbb{R})$ into $\mathbb{R}^n$. We observe that $0 \in \mathbb{R}^n$ is the unique fix point to this left action. Thus get $F_\sigma(u) = 0$. Therefore σ is the 0-section. □

Hopf fibration

Let $S^1 \to S^{2n-1} \to \mathbb{CP}^{n-1}$ be a Hopf fibration. It is well know that $S^{2n-1}(\mathbb{CP}^{n-1}, S^1)$ is a principal fiber bundle. We recall that $U(1) \cong S^1$. Let ϕ be the aplication of $U(1) \times \mathbb{C}^m$ into $\mathbb{C}^m$ given by

$$(g, (z_1, \ldots, z_m)) \to g \cdot (z_1, \ldots, z_m) = (gz_1, \ldots, gz_m). \qquad (15)$$

Clearly, ϕ is a left action of $U(1)$ into $\mathbb{C}^m$. Thus, we can consider $\mathbb{C}^m$ as standard fiber of associate fiber $E(\mathbb{CP}^{n-1}, \mathbb{C}^m, S^1, S^{2n-1})$, where $E = S^{2n-1} \times_{U(1)} \mathbb{C}^m$. We are considering the canonical scalar product $<, >$ on

$\mathbb{C}^n$ and the induced Riemannian metric g on $\mathbb{CP}^{n-1}$. Since $U(1)$ is invariant by $<,>$, there exists one and only one Riemannian metric $\hat{g}$ on E such that π_E is a Riemannian submersion from $(E, \hat{g})$ to (M, g) with totally geodesic fibers isometrics to $(N, <,>)$ (see for example [15]). From these assumptions and examples 2.1 and 2.2 wee see that hypotheses of Theorem 4.1 are holds.

Proposition 5.3 *Under conditions stated above, if σ is a harmonic section of π_E, then σ is the 0-section.*

Proof: We first observe that $(0, \ldots, 0)$ is the unique fix point to the left action (15). Since σ is harmonic section, from Theorem 4.1 we see that F_σ is constant map and $F_\sigma(p) = (0, \ldots, 0)$ for all $p \in S^{2n-1}$. Therefore σ is the 0-section. $\square$

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